BAULKHAM HILLS HIGH SCHOOL

2009

YEAR 12 HALF YEARLY EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

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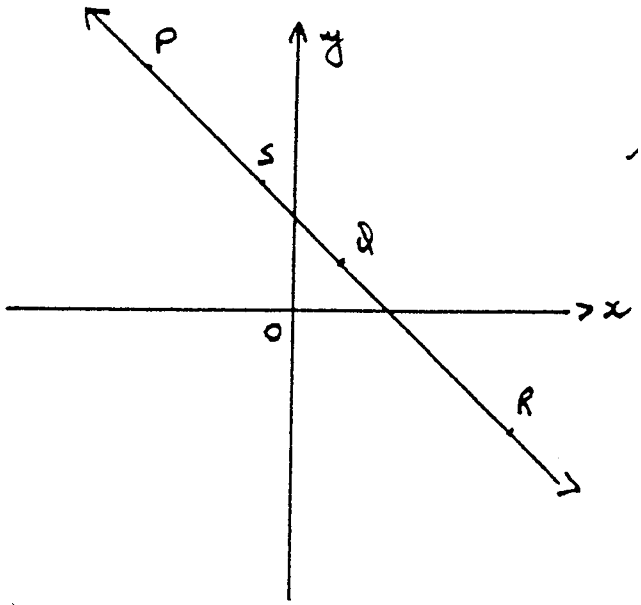
Answer each question on a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Marks

<u>Question 1</u> ***(12 marks)*** Use a *separate* piece of paper

a) Find the acute angle between the lines $2x + y = 15$ and $3x - y = 7$ *2*

b) *3*



P is the point $(-3,5)$.

$R(5,-3)$ divides PQ externally in the ratio 2:1. Find the coordinates of S which divides PQ internally in the ratio 2:1.

c) Find the Cartesian equation of the curve whose parametric coordinates are *2*

$$x = \cos 2t$$
$$y = \cos t$$

d) Evaluate $\lim_{x \to 0} \dfrac{\sin\left(\dfrac{x}{3}\right)}{3x}$ *2*

e) $x^2 - x - 2$ is a factor of $x^4 + 3x^3 + ax^2 - 2x - b$. Find the values of a and b. *3*

<u>Question 2</u> ***(12 marks)*** Use a *separate* piece of paper **_Marks_**

a) The equation $x^3 - 3x^2 + 4x + 2 = 0$ has roots α, β and γ. Find the values of;

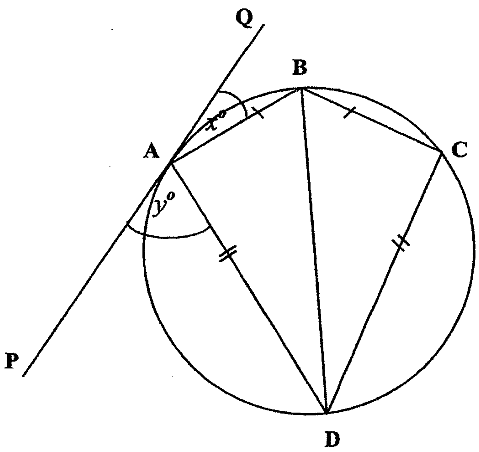
(i) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$ *2*

(ii) $\alpha^2 + \beta^2 + \gamma^2$ *2*

b) (i) On the same diagram sketch $y = x$ and $y = |x - 2|$ *2*

(ii) Hence or otherwise solve $\dfrac{|x-2|}{x} \geq 1$ *2*

c) The line *PQ* is a tangent to the circle at the point *A*.
AB = *BC* and *AD* = *DC*. $\angle QAB = x^\circ$ and $\angle PAD = y^\circ$



(i) Deduce that $x + y = 90^\circ$ *3*

(ii) Explain why *BD* is a diameter of the circle. *1*

<u>Question 3</u> ***(12 marks)*** Use a *separate* piece of paper

a) Find $\int \cos^2 2x dx$ *2*

b) If $\tan x = 2$, $0^\circ < x < 90^\circ$ and $\tan y = \dfrac{-1}{3}$, $90^\circ < y < 180^\circ$, find the exact value of $x + y$ *3*

c) Use $x = 0.5$ to find an approximation for the root of $\cos x = x$ using one application of Newton's Method, correct to 2 decimal places. *3*

d) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $A\sin(x + \alpha)$ *2*

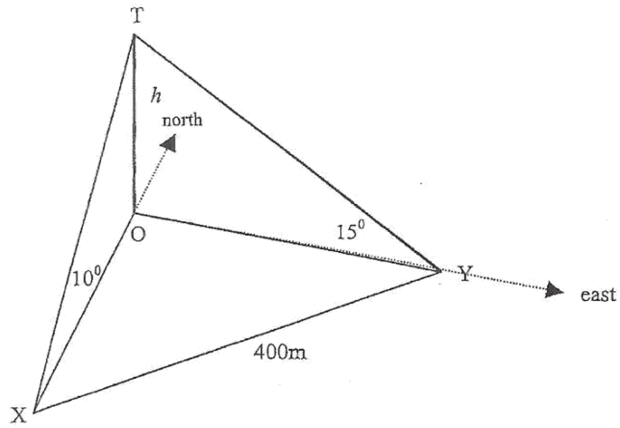
(ii) Hence, or otherwise, solve $\sin x + \sqrt{3}\cos x = 1$ for $0 \leq x \leq 2\pi$ *2*

Question 4 ***(12 marks)*** Use a *separate* piece of paper **Marks**

a) Show that $\dfrac{1+\cos 2\theta}{\sin 2\theta} \equiv \cot\theta$ *2*

b) A point moves along the curve $y = \dfrac{1}{x}$ such that the *x* value is changing at a rate of 2 units per second. *3*
At what rate is the *y* value decreasing when $x = 5$?

c) A surveyor at X observes a tower due north. The angle of elevation to the top of the tower is 10°. He then walks 400 metres to a position Y which is due east of the tower. The angle of elevation from Y to the top of the tower is 15°.



(i) Write equations for OX and OY in terms of *h*. *2*

(ii) Calculate *h* to the nearest metre. *3*

(iii) Find the bearing of X from Y, to the nearest degree. *2*

Question 5 ***(12 marks)*** Use a *separate* piece of paper

a) Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4. *3*

b) (i) Show that $\dfrac{d}{dx}\left(\tan^3 x\right) = 3\left(\sec^4 x - \sec^2 x\right)$ *2*

(ii) Hence find $\int \sec^4 x \, dx$ *2*

c) At time *t* minutes, the temperature T° of a body in a room of constant temperature 20° is decreasing according to the equation $\dfrac{dT}{dt} = -k(T-20)$ for some constant $k > 0$

(i) Verify that $T = 20 + Ae^{-kt}$, where *A* is constant, is a solution of the equation. *2*

(ii) The initial temperature of the body is 90° and it falls to 70° after 10 minutes. Find the temperature of the body after a further 5 minutes, correct to the nearest degree. *3*

Marks

Question 6 ***(12 marks)*** Use a *separate* piece of paper

a) $P\left(2ap, ap^2\right)$ is a point on the parabola $x^2 = 4ay$. The focus is $S\left(0, a\right)$.
PN is perpendicular to the tangent at *P*. *SN* is parallel to the tangent at *P*.

(i) Show that the equation of *PN* is $x + py = ap^3 + 2ap$ *2*

(ii) Show that the coordinates of *N* are $\left(ap, ap^2 + a\right)$ *3*

(iii) Find the Cartesian equation of the locus of *N*. *1*

b) Mayank borrows \$30000 at 9% per annum reducible interest calculated monthly.
The loan is to be repaid in 60 equal monthly instalments of \$M.

(i) Show that the amount A_n owing after *n* monthly repayments have been made is *2*

$$A_n = 30000\left(1.0075\right)^n - M\left(\frac{1.0075^n - 1}{0.0075}\right)$$

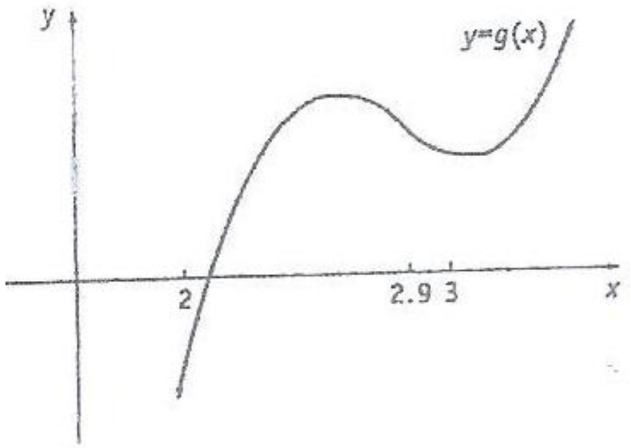
(ii) Show that the monthly repayments should be \$622.75 *1*

(iii) With the 12th repayment, Mayank pays an additional \$6000, so this repayment is \$6622.75. *3*

How many more repayments will be needed?

Question 7 ***(12 marks)*** Use a *separate* piece of paper

a) *2*



Consider the above graph of $y = g\left(x\right)$. There is a root between x = 2 and x = 3.
Explain why x = 2.9 is not a suitable first approximation for Newton's method to find this root.

Question 7 continues on page 6

Marks

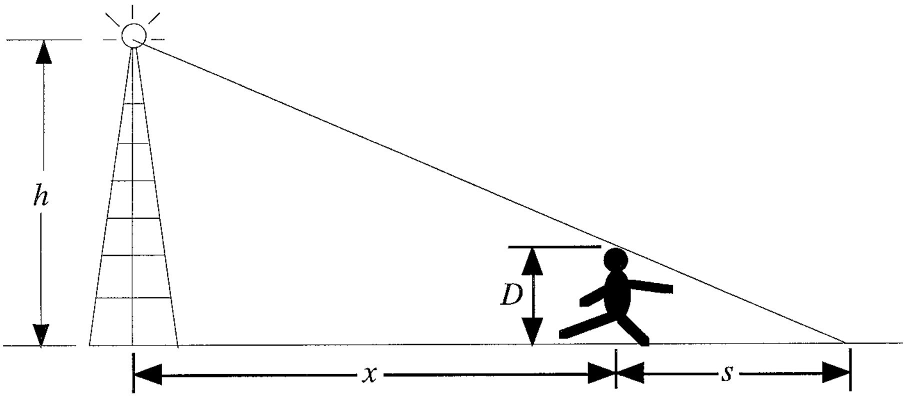
Question 7 ***(continued)***

b) (i) Use mathematical induction to prove for positive integral values of *n*; *3*

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\ldots+\frac{1}{n(n+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}$$

(ii) Hence, or otherwise, find $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+2)}$ *1*

c) Two lifeguards, David and Pamela, were having an argument on the beach. David bet Pamela \$50 that he could outrun his own shadow. That night they return to the beach so David could prove his argument.



In the diagram, *h* represents the height of the lighthouse, *D* represents David's height, *x* is David's displacement from the lighthouse and *s* is the length of David's shadow.
David runs with a velocity of *v* metres per second.

(i) Prove that $x = s\left(\frac{h}{D}-1\right)$. *2*

(ii) Hence, or otherwise, show that $\frac{ds}{dt}=\frac{Dv}{h-D}$ *2*

(iii) Assuming the lighthouse is 5 metres tall and David is 1.2 metres tall, show that $\frac{ds}{dt} < v$ *1*

(iv) David believed that he had proven his point and stated; "If the velocity of the shadow is always less than my velocity, then eventually I must overtake it." However Pamela saw a mistake in David's argument and refused to pay him \$50. What was David's mistake? *1*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; x \neq 0, \text{ if } \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log x, \quad x > 0$

2009 Extension 1 Half Yrly Solutions

Question 1 (12)

a) $2x+y=15 \Rightarrow m_1=-2$
$3x-y=7 \Rightarrow m_2=3$

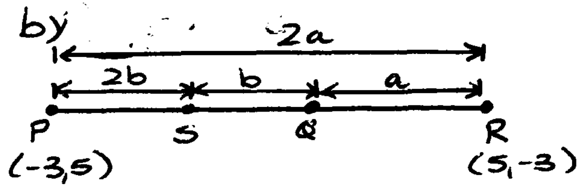
$\tan\alpha = \left|\dfrac{-2-3}{1+(-2)(3)}\right|$

$= \left|\dfrac{-5}{-5}\right|$

$= 1$

$\underline{\alpha = 45^\circ}$ (2)

b)



$a = 3b$

$\therefore$ S divides PR internally in the ratio

$2:4 = 1:2$

P(−3,5) R(5,−3)

1 : 2

$S = \left(\dfrac{-6+5}{3}, \dfrac{10-3}{3}\right)$

$= \underline{\left(-\dfrac{1}{3}, \dfrac{7}{3}\right)}$ (3)

c) $x = \cos 2t$ $\quad y = \cos t$

$x = 2\cos^2 t - 1$

$\underline{x = 2y^2 - 1}$ (2)

d) $\lim\limits_{x\to 0} \dfrac{\sin\left(\frac{x}{3}\right)}{3x}$

$= \lim\limits_{x\to 0} \dfrac{1}{9} \times \dfrac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}$

$= \underline{\dfrac{1}{9}}$ (2)

e) $x^2 - x - 2 = (x-2)(x+1)$

$\therefore P(2) = 0$ $\qquad P(-1) = 0$

$16+24+4a-4-b=0$ $\qquad 1-3+a+2-b=0$

$4a-b=-36$ $\qquad a-b=0$

$3a=-36$ $\qquad a=b$

$a=-12$

$\therefore \underline{a=-12,\ b=-12}$ (3)

Question 2 (12)

a) $x^3 - 3x^2 + 4x + 2 = 0$

(i) $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$

$= \dfrac{\gamma+\beta+\alpha}{\alpha\beta\gamma}$

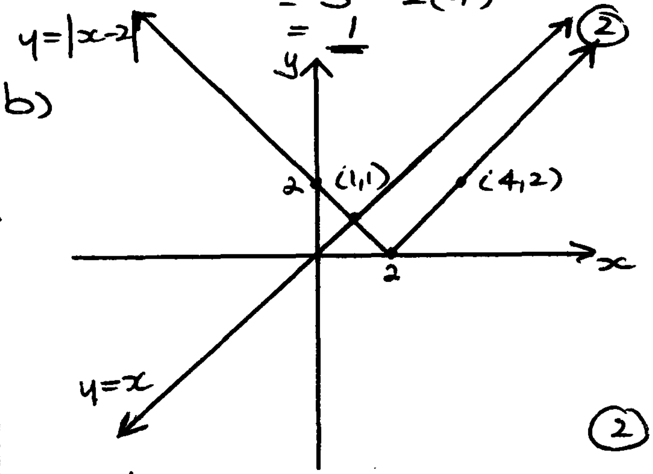
$= \underline{\dfrac{3}{-2}}$ (2)

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 3^2 - 2(4)$

$= \underline{1}$ (2)

b)



(2)

(ii) $\dfrac{|x-2|}{x} \geq 1$

If $x > 0$ $\qquad x < 0$

$|x-2| \geq x$ $\qquad |x-2| \leq x$,

$0 < x \leq 1$ $\qquad$ no solutions

$\therefore \underline{0 < x \leq 1}$ (2)

c) AB = CB (given)

AD = CD (given)

BD is common

$\therefore \triangle ABD \equiv \triangle CBD$ (SSS)

$\angle BAD = \angle BCD$ (matching ∠'s in ≡ △'s)

$\angle BAD + \angle BCD = 180^\circ$ (opposite ∠'s in cyclic quadrilateral)

$\therefore \angle BAD = \angle BCD = 90^\circ$

$x + y + 90^\circ = 180^\circ$ (straight ∠QAP)

$\underline{x + y = 90^\circ}$ (3)

(ii) $\angle BAD = 90^\circ$

$\therefore$ BD is diameter

(∠ in semicircle = 90°) (1)

Question 3 (12)

a) $\int \cos^2 2x \, dx$

$= \frac{1}{2}\int (1+\cos 4x)\,dx$

$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + c$

$= \frac{1}{2}x + \frac{1}{8}\sin 4x + c$ (2)

b) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{2 - \frac{1}{3}}{1 - (2)(-\frac{1}{3})}$

$= \frac{6-1}{3+2}$

$= 1$

$x+y = 45°$

however,

$0 < x < 90$ and $90 < y < 180$

$\therefore 90 < x+y < 270$

so $x+y$ is in 3rd quadrant as $\tan(x+y) > 0$

$\therefore x+y = 225°$ (3)

c) $f(x) = x - \cos x$

$f'(x) = 1 + \sin x$

$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}$

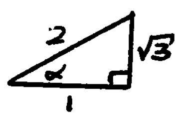
$x_1 = 0.5 - \frac{0.5 - \cos 0.5}{1 + \sin 0.5}$

$= 0.7552224171$

$= 0.76$ (to 2dp) (3)

d) (i) $\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$



$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$ (2)

(ii) $\sin x + \sqrt{3}\cos x = 1$

$2\sin\left(x+\frac{\pi}{3}\right) = 1$

$\sin\left(x+\frac{\pi}{3}\right) = \frac{1}{2}$

$x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = -\frac{\pi}{6}, \frac{\pi}{2}$

$\therefore x = \frac{11\pi}{6}, \frac{\pi}{2}$ (2)

Question 4 (12)

a) $\frac{1+\cos 2\theta}{\sin 2\theta} = \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$

$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$

$= \frac{\cos\theta}{\sin\theta}$

$= \cot\theta$ (2)

b) $y = \frac{1}{x}$

$\frac{dy}{dt} = -\frac{1}{x^2} \cdot \frac{dx}{dt}$

$= \frac{-2}{x^2}$

when $x = 5$, $\frac{dy}{dt} = \frac{-2}{25}$

$\therefore$ y coordinate is decreasing at a rate of $\frac{2}{25}$ units/s (3)

c) $\frac{OX}{h} = \tan 80°$ $\qquad$ $\frac{OY}{h} = \tan 75°$

$OX = h\tan 80°$ $\qquad$ $OY = h\tan 75°$ (2)

(ii) $OX^2 + OY^2 = XY^2$

$h^2\tan^2 80° + h^2\tan^2 75° = 400^2$

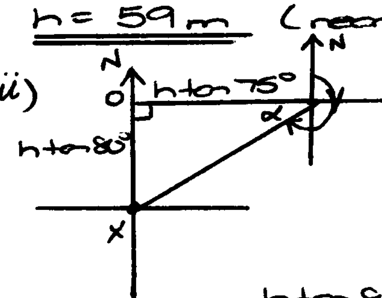
$h^2 = \frac{400^2}{\tan^2 80° + \tan^2 75°}$

$h = \frac{400}{\sqrt{\tan^2 80° + \tan^2 75°}}$

$h = 58.91812363$ (3)

$h = 59\,m$ (nearest metre)

(iii)



$\tan\alpha = \frac{h\tan 80°}{h\tan 75°}$

$\alpha = 56.65263461$

$\alpha = 57°$ (nearest degree)

$\therefore$ Bearing $= 270 - 57$

$= 213°$ (2)

Question 5 (12)

a) $3x^3 - 17x^2 - 8x + 12 = 0$

$\alpha\beta\gamma = -4$

$4\gamma = -4$

$\gamma = -1$

$(x+1)(3x^2 - 20x + 12) = 0$

$(x+1)(x-6)(3x-2) = 0$

$x = -1, 6, \frac{2}{3}$ (3)

b) $\frac{d}{dx}(\tan^3 x)$

$= 3\tan^2 x \sec^2 x$

$= 3(\sec^2 x - 1)\sec^2 x$

$= 3(\sec^4 x - \sec^2 x)$ (2)

(ii) $\tan^3 x = 3\int(\sec^4 x - \sec^2 x)\,dx$

$3\int \sec^4 x\,dx = \tan^3 x + 3\int \sec^2 x\,dx$

$= \tan^3 x + 3\tan x + c$

$\therefore \int \sec^4 x\,dx = \frac{1}{3}\tan^3 x + \tan x + c$ (2)

c) $T = 20 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(20 + Ae^{-kt} - 20)$

$= -k(T - 20)$ (2)

(ii) when $t = 0$, $T = 90$

$\therefore 90 = 20 + A$

$A = 70$

$T = 20 + 70e^{-kt}$

when $t = 10$, $T = 70$

$70 = 20 + 70e^{-10k}$

$50 = 70e^{-10k}$

$e^{-10k} = \frac{5}{7}$

$-10k = \log\frac{5}{7}$

$k = -\frac{1}{10}\log\frac{5}{7}$

$(= \frac{1}{10}\log\frac{7}{5})$

when $t = 15$,

$T = 20 + 70e^{-15k}$

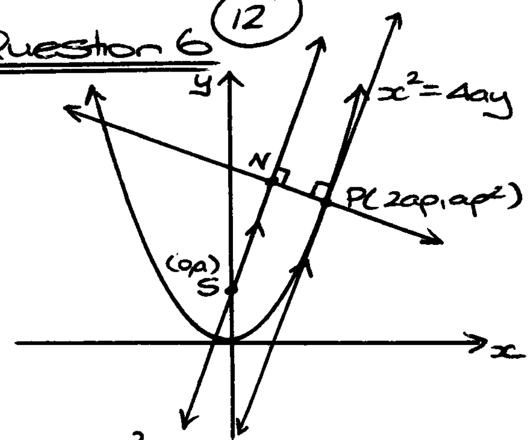
$= 20 + 70\left(\frac{5}{7}\right)^{\frac{15}{10}}$

$= 62.25771274$ (3)

$= 62°$ (nearest degree)

Question 6 (12)

a)



(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at P, $\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$ (2)

(ii) NS: $m_{NS} = p$

$y = px + a$

$x + p^2x + ap = ap^3 + 2ap$

$(p^2+1)x = ap^3 + ap$

$= ap(p^2+1)$

$x = ap$

$\therefore y = ap^2 + a$ (3)

N has coordinates $(ap, ap^2 + a)$

(iii) $x = ap$

$p = \frac{x}{a}$

$y = ap^2 + a$

$= a\left(\frac{x}{a}\right)^2 + a$

$y = \frac{x^2}{a} + a$ (1)

b) Initial loan borrowed for n months $= 30000(1.0075)^n$

1st repayment invested for (n−1) months $= M(1.0075)^{n-1}$

2nd repayment invested for (n−2) months $= M(1.0075)^{n-2}$

⋮

last repayment invested for 0 months $= M$

A_n = (principal + interest) − (repayments + interest)

$A_n = 30000(1.0075)^n - M(1 + 1.0075 + 1.0075^2 + \ldots + 1.0075^{n-1})$

$a = 1$, $r = 1.0075$, $n = n$

$$A_n = 30000(1.0075)^n - M\left[\frac{1.0075^n - 1}{0.0075}\right]$$ (2)

(ii) $A_{60} = 0$

$$0 = 30000(1.0075)^{60} - M\left[\frac{1.0075^{60} - 1}{0.0075}\right]$$

$$M = \frac{30000(1.0075)^{60}(0.0075)}{1.0075^{60} - 1}$$

$= 622.7506568$

$= \$622.75$ (1)

(iii) $$A_{12} = 30000(1.0075)^{12} - 622.75\left[\frac{1.0075^{12} - 1}{0.0075}\right] + 6000$$

$= 19025.10753$

$= \$19025.11$

So

$$A_n = 19025.11(1.0075)^n - 622.75\left[\frac{1.0075^n - 1}{0.0075}\right]$$

But $A_n = 0$

$$0 = 19025.11(1.0075)^n - 83033.\dot{3}(1.0075^n - 1)$$

$$64008.22\dot{3}(1.0075)^n = 83033.\dot{3}$$

$$(1.0075)^n = 1.297229153\ldots$$

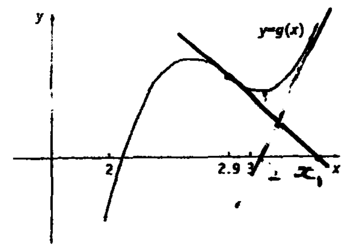
$$n = \frac{\log 1.297229153\ldots}{\log 1.0075}$$

$= 34.82736244$

∴ 35 more repayments are needed (3)

Question 7 (12)

a)



(2)

As 2.9 is on the opposite side of the stationary point to the root, the tangent will cut the x-axis further from the root than the original approximation.

b) Prove true for $n = 1$

LHS $= \frac{1}{1\times 3} = \frac{1}{3}$ RHS $= \frac{1(3+5)}{4(2)(3)} = \frac{1}{3}$

∴ LHS = RHS

Hence the result is true for $n=1$

Assume the result is true for $n = k$, where k is a positive integer

ie $$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \ldots + \frac{1}{k(k+2)} = \frac{k(3k+5)}{4(k+1)(k+2)}$$

Prove the result is true for $n = k+1$

ie Prove

$$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \ldots + \frac{1}{(k+1)(k+3)} = \frac{(k+1)(3k+8)}{4(k+2)(k+3)}$$

PROOF:

$$\frac{1}{1\times 3} + \frac{1}{2\times 4} + \ldots + \frac{1}{k(k+2)} + \frac{1}{(k+1)(k+3)}$$

$$= \frac{k(3k+5)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+3)}$$

$$= \frac{k(3k+5)(k+3) + 4(k+2)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{3k^3 + 14k^2 + 19k + 8}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(3k+8)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(3k+8)}{4(k+2)(k+3)}$$

Hence the result is true for $n = k+1$ if it is true for $n = k$

Since the result is true for $n=1$, then the result is true for all positive integral values of n by induction. (3)

(ii) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+2)}$

$= \lim_{n\to\infty} \frac{n(3n+5)}{4(n+1)(n+2)}$

$= \frac{3}{4}$ (1)

c) (i) $\frac{x+s}{h} = \frac{s}{D}$

(ratio of sides in ||| △'s)

$x+s = \frac{sh}{D}$

$x = \frac{sh}{D} - s$

$x = s\left(\frac{h}{D} - 1\right)$ (2)

(ii) $\frac{dx}{dt} = \frac{ds}{dt}\left(\frac{h}{D} - 1\right)$

$V = \frac{ds}{dt}\left(\frac{h}{D} - 1\right)$

$= \frac{ds}{dt}\left(\frac{h-D}{D}\right)$

$\frac{ds}{dt} = \frac{DV}{h-D}$ (2)

(iii) $\frac{ds}{dt} = \frac{1.2V}{5-1.2}$

$= \frac{1.2V}{3.8}$

$= \frac{6}{19}V$

$\therefore \frac{ds}{dt} < V$ (1)

(iv) The mistake is that $\frac{ds}{dt}$ measures the change in the length of the shadow NOT the velocity of the shadow. (1)